NORTH-HOLLAND

Rank 5 Association Schemes and Triality

D. G. Higman
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109

Dedicated to J. J. Seidel

Submitted by Willem Haemers

ABSTRACT

Symmetric association schemes having a parabolic of rank 3 and corank 2 satisfying a certain uniformity condition can be interpreted on the one hand as linked strongly regular designs (analogous to linked symmetric designs) and on the other as geometries. Classical triality associated with the groups $O_8^+(q)$ provides a family of examples with rank 3, corank 2 parabolics and sporadic examples are associated with the groups $L_3(4)$, $U_6(2)$ and $U_3(5)$. The triality examples and the $U_3(5)$ example are flag-transitive viewed as geometries. Results include a characterization on the level of parameters of the triality schemes leading to a characterization of the schemes using results of Cameron and Drake, and a proof (as a Cayley exercise) of simple connectivity of the example associated with $U_3(5)$.

0. INTRODUCTION

Motivated by examples associated with classical triality and related group-theoretical phenomena, we investigate imprimitive, symmetric rank 5 association schemes. As indicated in Section 1, the problem reduces to the consideration of three classes I, II, III according to the possibilities for the ranks of certain residues and quotients. Sections 2 through 8 are concerned with class I, which contains the examples equivalent to symmetric strongly regular designs and the examples related to triality. The intersection matrices and character-multiplicity tables are described in Section 2. In Section 4 we observe that *uniform* class I schemes as defined in Section 3 are equivalent to

LINEAR ALGEBRA AND ITS APPLICATIONS 226–228:197–222 (1995)

0024-3795/95/$9.50
SSDI 0024-3795(95)00102-W

a class of geometries which we refer to as *uniformly linked strongly regular designs*. This equivalence is analogous to the equivalence of certain imprimitive, symmetric association schemes of rank 4 with linked symmetric 2-designs ([PJC], [YC]). The family of examples provided by triality and three additional group-theoretic examples are described in Section 5. The geometries of the triality examples and one additional example associated with the group $U_3(5)$ are flag-transitive. A classification on the level of parameters is given in Section 6 for the triality examples. In Section 7 the geometry of the $U_3(5)$ example is shown to be simply connected by showing that $U_3(5)$ is the universal cover of the relevant amalgam. The distance regular schemes in class I are considered in Section 8. The intersection matrices and character-multiplicity tables for the classes II and III are described in Sections 9 and 11, and the intersection of classes I and II is determined on the level of parameters in Section 10. Class III is disjoint from classes I and II. We use [CAYLEY] to examine group-theoretic examples. In particular, the determination of the universal cover of the amalgam in Section 7 is done as a Cayley exercise. Maple V is used constantly for parameter computations. The book [BCN] is a very convenient reference, e.g., for association schemes and for additional references. Our conventions of terminology and notation are essentially those of [DGH1].

1. IMPRIMITIVE ASSOCIATION SCHEMES OF RANK 5

An association scheme $\mathfrak{X}$ of rank r can be specified as pair (Ω, δ), where Ω is a finite set and $\delta : \Omega \times \Omega \to \Lambda = \{0, 1, 2, \dots, r-1\}$ is a surjective map such that the fibers $\delta_i = \delta^{-1}(i)$, $i \in \Lambda$, of δ, viewed as binary relations on Ω, satisfy the usual axioms (see for example [BCN] or [DGH1]; our notational and terminological conventions essentially follow [DGH1]). The notation will be chosen so that $\delta_0 = \mathrm{diag}(\Omega \times \Omega)$. The scheme $\mathfrak{X}$ is *symmetric* if δ is symmetric, i.e., if the relations δ_i are symmetric. Then the Bose-Mesner algebra is commutative, and the graphs $\Gamma_i = (\Omega, \delta_i)$, $i \neq 0$, are simple graphs. The δ_i are the *basic relations* and the Γ_i are the *basic graphs*. An equivalence relation on Ω of the form $\delta^{-1}(S)$, $S \subseteq \Lambda$, will be called a *parabolic*, and δ_0 and $\Omega \times \Omega$ are the *trivial parabolics*. The *rank* $|S|$ of E is equal to the rank of the association scheme induced on each equivalence class modulo E and the *corank* of E is the rank of the quotient scheme modulo E ([DGH3]). The schemes induced on the equivalence classes will be referred to as the *residues modulo E*. The sum of the rank and corank of E is at most $r + 1$, with equality if and only if $\mathfrak{X}$ is the wreath product of the residues modulo E by the quotient modulo E (or briefly, if and only if $\mathfrak{X}$ is a wreath product modulo E) ([DGH3])). A scheme is *primitive* if there are no nontrivial parabolics, and *imprimitive* otherwise.

We are interested here in imprimitive, symmetric association schemes $\mathfrak{X}$ of rank 5 containing a parabolic E such that $\mathfrak{X}$ is not a wreath product modulo E. Such parabolics E fall into three classes which can be indicated as follows:

class of E	I	II	III
rank of E	3	2	2
corank of E	2	3	2.

Let us say that $\mathfrak{X}$ is in class I, II or III if it contains a parabolic of class I, II or III, respectively. It turns out that classes I and II of schemes have nontrivial intersection while each of these classes meets class III trivially. We consider these classes of schemes in turn, with the main emphasis on class I, which contains schemes associated with triality.

2. CLASS I SCHEMES

Suppose that $\mathfrak{X} = (\Omega, \delta)$ is a class I scheme with class I parabolic E. We may assume that $E = \delta_0 \cup \delta_1 \cup \delta_2$, and denote the equivalence classes modulo E by Ω_α, $\alpha \in \Sigma = \{0, 1, \ldots, t\}$, $t \geqslant 1$. The residues modulo E, i.e., the schemes induced on the Ω_α, are symmetric rank 3 association schemes all having the same intersection matrices and character-multiplicity table, which we write as

$$\underline{m}_1 = \begin{pmatrix} 0 & 1 & 0 \\ k & \lambda & \mu \\ 0 & k-\lambda-1 & k-\mu \end{pmatrix},$$

$$\underline{m}_2 \cong \begin{pmatrix} 0 & 0 & 1 \\ 0 & k-\lambda-1 & k-\mu \\ l & \tilde{\mu} & \tilde{\lambda} \end{pmatrix}, \quad \begin{array}{c} \begin{pmatrix} 1 & k & l \\ 1 & r & -(r+1) \\ 1 & s & -(s+1) \end{pmatrix} \end{array} \begin{array}{l} 1 \\ f. \\ g \end{array}$$

Two elements x, $y \in \Omega$ will be called *adjacent* if $(x, y) \in \delta_1$ and *incident* if $(x, y) \in \delta_3$. We write $x * y$ to mean that x and y are incident and define the *type* $\tau(x)$ of x to be α if $x \in \Omega_\alpha$. In this way, we can view Ω as the set of varieties of the geometry $(\Omega, *, \tau)$, which we refer to as the geometry *associated* with the given scheme. (Of course, we have made an arbitrary choice of δ_3 over δ_4 in defining incidence in this geometry—the opposite choice defines the complementary geometry. We remark that the residues modulo E as defined above and the residues of the geometry in the usual sense are different objects.) The total number of varieties is $|\Omega| = (t+1)v$. If α and β are distinct elements of Σ and x has type α, then the number of varieties of type β incident with x is $S = 1 + p_{31}^3 + p_{32}^3$ and the total number of elements incident with x is tS. The valencies for the class I scheme are $v_0 = 1$, $v_1 = k$, $v_2 = l$, $v_3 = tS$, and $v_4 = t(v - S)$. The first two intersection matrices for the rank 5 scheme have the form

$$M_1 = \begin{pmatrix} m_1 & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} m_2 & 0 \\ 0 & B \end{pmatrix},$$

where

$$A = \begin{pmatrix} N & P \\ k - N & k - P \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} S - N - 1 & S - P \\ l - S + N + 1 & l - S + P \end{pmatrix}.$$

Here N is the number of varieties incident with x and adjacent to y, where x and y are incident varieties, and P is the corresponding number for nonincident varieties of different types. By a standard formula for intersection numbers, we have

$$(k - N)S = P(v - S), \qquad 0 < N < k. \tag{2.1}$$

The eigenvalues of A are k with eigenvector $\begin{pmatrix} S \\ v - S \end{pmatrix}$ and $N - P$ with eigenvector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Hence $N - P \in \{k, r, s\}$, and since $0 \leqslant N \leqslant k$, we may assume that

$$N - P = r \tag{2.2}$$

and then

$$P = \frac{S(k - r)}{v}. \tag{2.3}$$

The eigenspaces for M_1 and M_2 are $\langle \underline{a}, \underline{b} \rangle$ for eigenvalue k, $\langle \underline{c}, \underline{d} \rangle$ for eigenvalue r, and $\langle \underline{e} \rangle$ for eigenvalue s, where

$$\underline{a} = (1, k, l, tS, t(v - S))^t, \qquad \underline{b} = (0, 0, 0, S, v - S)^t,$$

$$\underline{c} = (1, r, -(r + 1), 0, 0)^t, \qquad \underline{d} = (0, 0, 0, 1, -1)^t,$$

and

$$\underline{e} = (1, s, -(s+1), 0, 0)^t.$$

The remaining intersection matrices have the form

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & N & P \\ 0 & 0 & 0 & S-N-1 & S-P \\ tS & \dfrac{NtS}{k} & \dfrac{(S-N-1)tS}{l} & \rho & \sigma \\ 0 & \dfrac{Pt(v-S)}{k} & \dfrac{(S-P)t(v-S)}{l} & (t-1)S-\rho & (t-1)S-\sigma \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & k-N & k-P \\ 0 & 0 & 0 & l-S+N+1 & l-S+P \\ 0 & \dfrac{Pt(v-S)}{k} & \dfrac{(S-P)t(v-S)}{l} & (t-1)S-\rho & (t-1)S-\sigma \\ t(v-S) & \dfrac{(k-P)t(v-S)}{k} & \dfrac{(l-S+P)t(v-S)}{l} & (t-1)(v-2S)+\rho & (t-1)(v-2S)+\sigma \end{pmatrix}.$$

Here ρ is the number of varieties incident with two incident varieties x and y and σ is the corresponding number for nonincident x and y of different types. Again by a standard relation on intersection numbers,

$$[(t-1)S-\rho]S = \sigma(v-S). \tag{2.4}$$

The equation $M_3^2 = tSI + (NtS/k)M_1 + \{[(S-N-1)tS]/l\}M_2 + \rho M_3 + \sigma M_4$ applied to the (1, 1)-entry gives

$$N^2 + P(k-N) = k + N\lambda + (S-N-1)\mu. \tag{2.5}$$

The matrix M_3 must have eigenvectors of the form $(1, k, l, Sx, (v-S)x)^t$ with eigenvalue Sx. Multiplying this vector by the next-to-last row of M_3 gives the equation $tS + NtS + (S-N-1)tS + \rho Sx + \sigma(v-S)x = S^2x^2$. Using (2.4), this reduces to $x^2 - (t-1)x - t = 0$ with roots $x = t, -1$. Also M_3 has eigenvectors of the form $(1, r, -(r+1), x, -x)$ with eigenvalue x. Multiplying this vector by the next-to-last row of M_3 gives the equation

$$x^2 - (\rho-\sigma)x - C = 0, \qquad C = tS\left(1 + \frac{Nr}{k} - \frac{(S-N-1)(r+1)}{l}\right). \tag{2.6}$$

It now follows that the character-multiplicity table of our class I scheme $\mathfrak{X}$ is

$$\left(\begin{array}{ccccc} 1 & k & l & tS & t(v-S) \\ 1 & k & l & -S & -(v-S) \\ 1 & r & -(r+1) & x_1 & -x_1 \\ 1 & r & -(r+1) & x_2 & -x_2 \\ 1 & s & -(s+1) & 0 & 0 \end{array}\right)\begin{array}{l} 1 \\ t \\ z_1 \\ z_2 \\ (t+1)g \end{array},$$

where x_1 and x_2 the roots of the quadratic (2.6). The orthogonality relations give

$$f = -\frac{tS(v-S)}{x_1 x_2} \tag{2.7}$$

and

$$z_1 = \frac{(t+1)fx_2}{x_2 - x_1}, \qquad z_2 = \frac{(t+1)fx_1}{x_1 - x_2}, \qquad z_1 + z_2 = (t+1)f. \tag{2.8}$$

From the equation for M_3^2, we obtain

$$[(S - N - 1)k - Nl]s = (l - S + N + 1)k. \tag{2.9}$$

If x_1 and x_2 are irrational, then $z_1 = z_2 = [(t+1)f]/2$, so $x_1^2 = x_2^2 = C$ and $\rho = \sigma = [(t+1)S^2]/v$. Otherwise, x_1 and x_2 must be integers.

The intersection matrices (equivalently, the character table) are determined by the srg parameters v, k, r, s, the number t of residues and the parameters S and ρ.

Let $K = 1 + r^3/k^2 - [(r+1)^3]/l^2$, so that $K \geqslant 0$ by the rank 3 Krein conditions, and let

$$L = \frac{x_2^3(v^2 - 2S)}{t^2S^2(v-S)^2}, \qquad M = \frac{x_1^2 x_2(v^2 - 2S)}{t^2S^2(v-S)^2}.$$

Then the rank 5 Krein conditions reduce to $K + L \geqslant 0$ and $K + M \geqslant 0$, where x_1 is the positive and x_2 the negative root of (2.6).

A class I scheme with notation as above fuses to the rank 4 scheme $(\Omega, \delta_1, \delta_2, \delta_3 \cup \delta_3)$, which is a wreath product.

Let us suppose that the class I scheme $\mathfrak{X}$ contains a nontrivial parabolic $F \neq E$. If F has rank 2 and $F \cap E = \delta_0$, then we may assume that $F = \delta_0 \cup \delta_2$. But then $M_3^2 \in \langle I, M_3 \rangle$, and hence $N = 0$ and $S - N - 1 = 0$, which is impossible. Thus if F has rank 2, then $F \subseteq E$ and either F has corank 3 and $\mathfrak{X}$ is in the intersection of classes I and II, or F has corank 4 and $\mathfrak{X}$ is a wreath product modulo F. The intersection of classes I and II will be considered further in Section 10. In a similar way, we see that if F has rank 2, then $E \wedge F$ must have rank 2 and corank 4, so $\mathfrak{X}$ is a wreath product modulo $E \wedge F$. Of course if F has rank 4, then $E \subseteq F$ and $\mathfrak{X}$ is a wreath product modulo F.

3. UNIFORM CLASS I SCHEMES

A class I scheme $\mathfrak{X}$ with class I parabolic $E = \delta_0 \cup \delta_1 \cup \delta_2$ having primitive residues will be called *uniform* if it satisfies the following two conditions:

(1) If x, y are distinct varieties of the same type α, then the number of varieties of any given type $\beta \neq \alpha$ incident with x and y is $\underline{a}$ if x and y are adjacent and b otherwise, where a and b are constants independent of α and β.

(2) If α, β, γ are distinct elements of Σ, x is a variety of type α, and y is a variety of type β, then the number of varieties of type γ incident with x and y is c if x and y are incident and d otherwise, where c and d are constants independent of α, β and γ.

Of course (2) is vacuous in case $t = 1$. Observe that if we are given a uniform class I scheme with $t + 1$ residues, then we obtain one with $t' + 1$ residues for each $1 \leqslant t' < t$ simply by discarding any $t - t'$ residues.

For a uniform class I scheme, we have $p_{33}^1 = NtS/k = ta$, so $NS = ak$, and $p_{33}^2 = [(S - N - 1)tS]/l = tb$, so $(S - N - 1)S = bl$, and $p_{33}^3 = \rho = (t - 1)c$ and $p_{33}^4 = \sigma = (t - 1)d$.

Let T_m denote the $m \times m$ matrix with diagonal entries 3 and all other entries 2. The function $f : \Omega \times \Omega \to \Lambda \times \Sigma \times \Sigma$, $f(x, y) = (\delta(x, y), \tau(x), \tau(y))$, defines a coherent configuration on Ω of type T_3 with fibers Ω_α, $\alpha \in \Sigma$, if and only if $\mathfrak{X}$ is uniform. (For coherent configurations, we use the notational conventions of [DGH1].) Because of our assumption that the residues of the scheme are primitive, the geometry induced on any two distinct fibers is a symmetric strongly regular design with the parameters having the same significance as in [DGH2]. In particular, we must have $a \neq b$, $S(S - 1) = ak + bl$ and $S - b = -(a - b)s$.

Assume that $\mathfrak{X}$ is uniform with $t \geqslant 2$, and let α, β, γ be distinct elements of Σ. Then

$$M^{\alpha}_{(4,\beta,\gamma)} = M^{\alpha}_{(4,\gamma,\beta)} = \begin{pmatrix} c & d \\ S-c & S-d \end{pmatrix}$$

and therefore

$$\begin{pmatrix} c & d \\ S-c & S-d \end{pmatrix}^2 = \sum_{i=0}^{3} p^{(i,\beta,\beta)}_{(4,\beta,\gamma)(4,\gamma,\beta)} = SI + a\begin{pmatrix} N & P \\ k-N & k-P \end{pmatrix}$$

$$+ b\begin{pmatrix} S-N-1 & S-P \\ l-S+N+1 & l-S+P \end{pmatrix}.$$

Hence

$$c^2 + d(S-c) = S + aN + b(S-N-1) \tag{3.1}$$

and

$$cd + d(S-d) = aP + b(S-P), \tag{3.2}$$

from which we obtain $(c-d)^2 = (a-b)(r-s)$. That is,

$$c - d = \varepsilon u, \text{ where } \varepsilon = \pm 1 \text{ and } u^2 = (a-b)(r-s),\ u > 0. \tag{3.3}$$

But by (2.4),

$$d(v-S) = S(S-c). \tag{3.4}$$

Hence

$$c = d + \varepsilon u, \qquad d = \frac{S(S-\varepsilon u)}{v}. \tag{3.5}$$

Thus for a uniform class I scheme, we have

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & N & P \\ 0 & 0 & 0 & S-N-1 & S-P \\ tS & ta & tb & (t-1)c & (t-1)d \\ 0 & t(S-a) & t(S-b) & (t-1)(S-c) & (t-1)(S-d) \end{pmatrix},$$

and

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & k-N & k-P \\ 0 & 0 & 0 & v-S-k+N & v-S-k+P-1 \\ 0 & t(S-a) & t(S-b) & (t-1)(S-c) & (t-1)(S-d) \\ t(v-S) & t(v-2S+a) & t(v-2S+b) & (t-1)(v-2S+c) & (t-1)(v-2S+d) \end{pmatrix}.$$

Note that $C = t(a-b)(r-s) = tu^2$ and $\rho - \sigma = (t-1)(c-d) = (t-1)\varepsilon u$, so (2.5) becomes $x^2 - (t-1)\varepsilon ux + tu^2 = 0$, and therefore $x_1 = -\varepsilon u$, $x_2 = t\varepsilon u$. Therefore $S(v-S) = u^2 f$ and $z_1 = f$ and $z_2 = tf$. Thus the character-multiplicity table is

$$\begin{array}{c} \begin{pmatrix} 1 & k & l & tS & t(v-S) \\ 1 & k & l & -S & -(v-S) \\ 1 & r & -(r+1) & \varepsilon tu & -\varepsilon tu \\ 1 & r & -(r+1) & -\varepsilon u & \varepsilon u \\ 1 & s & -(s+1) & 0 & 0 \end{pmatrix} \begin{array}{l} 1 \\ t \\ f \\ tf \\ (t+1)g \end{array} \end{array}.$$

Let $K = 1 + r^3/k^2 - (r+1)^3/l^2$ as above; then the Krein conditions become

$$\text{if } c > d, \quad \text{then } K \geqslant \frac{t(v-2S)}{uf^2}, \quad \text{and} \tag{3.6}$$

$$\text{if } c < d, \quad \text{then } t \leqslant \frac{Kuf^2}{v-2S}. \tag{3.7}$$

We do not know a bound on t in case $c > d$. In all the examples we have at this time with $t > 1$ (see Section 5), the number of residues is three and $c > d$.

Fix a residue Ω_0 and refer to the varieties in Ω_0 as *points*. For a variety x of type $\neq 0$, let $[x]$ be the set of points incident with x. Suppose that y is a variety $\neq x$ of type $\neq 0$ such that $[x] = [y]$. If x and y have the same type, then $S = |[x] \cap [y]| = a$ or b according as x and y are adjacent or not. But then $s = -1$ or $a = b$, both of which are excluded. If x and y are not of the same type, then $S = c$ or d according as x and y are incident or not. If $S = c$, then by (3.4) above, $d = 0$ since $v = S$ is impossible. But then by (3.2), $aP + b(S - P) = 0$, which is impossible. Hence $S = d$, which gives a contradiction for the complementary geometry (for which incidence is δ_4 instead of δ_3). Thus in all cases, if x and y are distinct varieties of type $\neq 0$, then $[x] \neq [y]$. This means that we can identify each variety x of type $\alpha \neq 0$ with the set $[x]$ points incident with x, which we refer to as a *block* of type α. If x and y are varieties of type $\neq 0$, then

$$[x] \cap [y] = S, a, b, c, d \text{ according as } x \text{ and } y \text{ are}$$
$$\text{adjacent, nonadjacent, incident or nonincident.}$$

We know that $a \neq b$ and $c \neq d$ for uniform class I schemes, and a, b, c, d are distinct in the examples that we know. But there are feasible parameter sets for which these numbers are not all distinct.

4. UNIFORMLY LINKED STRONGLY REGULAR DESIGNS

Suppose a geometry $(\Omega, *, \tau)$, $\tau : \Omega \to \Sigma = \{0, 1, 2, \ldots, t\}$, $t \geqslant 1$. Let $\Omega_\alpha = \tau^{-1}(\alpha)$, $\alpha \in \Sigma$. We refer to the geometry as a *system of uniformly linked strongly regular designs* if there exist constants v, S, a, b, N, c, d, with $a \neq b$, such that the following conditions (i) through (v) hold:

(i) $|\Omega_\alpha| = v$.

(ii) If α, β are distinct elements of Σ and x is a variety of type α, then the number of varieties of type β incident with x is S.

(iii) If x and y are distinct varieties of type α, then for $\beta \in \Sigma$, $\beta \neq \alpha$, the number of varieties of type β incident with x is a or b, independent of β.

Given distinct types α, β, two varieties x and y of type α will be called *adjacent* if the number of varieties of type β incident with x and y is a. Condition (iii) implies that this relation of adjacency on Ω_α is independent of β.

(iv) If x and y are incident varieties, then the number of varieties adjacent to x and incident with y is N.

(v) Given distinct types α, β, γ and varieties x, y of type α and β, respectively, then the number of varieties of type β incident with x and y is c if x and y are incident and d otherwise.

The geometry associated with a uniform class I scheme is a system of uniformly linked strongly regular designs, and we claim that, given such a system, there is a uniform class I scheme (Ω, δ) for which $(\Omega, _*, \tau)$ is the associated geometry. Namely, define $\delta : \Omega \times \Omega \to \{0, 1, 2, 3, 4\}$ by $\delta(x, y) =$ 0, 1, 2, 3, or 4 according as x and y are equal, adjacent, of the same type but not adjacent, incident, or of different type but not incident. The problem is to verify that (Ω, δ) is an association scheme. First, according to [DGH2], conditions (i), (ii), (iii) and (iv) imply that the geometry induced on the union of any two of the Ω_α is a strongly regular design with parameters v, S, a, b, N, and (iii) implies that the rank 3 schemes on the Ω_α are all the same. Now assume that $t > 1$, and consider the configuration induced on the partition of Ω into the Ω_α. The additional condition (v) implies that this is a coherent configuration of type T_{t+1}, and from this the claim follows easily.

5. EXAMPLES OF UNIFORM CLASS I SCHEMES

In the case $t = 1$, such schemes are equivalent to symmetric strongly regular designs, and we refer to [DGH2] for examples of these. The examples that we know with $t > 1$ all have $t = 2$ and are instances of the following situation. There is given a group $\Gamma = G \cdot \langle \sigma, \vartheta \rangle$ which is a semidirect product of a subgroup and a subgroup $\langle \sigma, \vartheta \rangle \cong S_3$ with $\sigma^2 = \vartheta^3 = 1$, $\sigma\vartheta = \vartheta^2\sigma$, and there is given a transitive (left) action of Γ on a set Ω. Fixing $1 \in \Omega$, we can identify Ω with the set Γ/H of left cosets of $\Gamma \bmod H$, where H is the stabilizer in Γ of $1 \in \Omega$. In the examples, the action of G on Ω is of type T_3 and thus affords a coherent configuration of this type with the orbits Ω_1, Ω_2, Ω_3 of G in Ω as fibers, and $\langle \sigma, \vartheta \rangle$ acts as S_3 on this set of orbits. The action of Γ on Ω affords a uniform class I scheme with the Ω_i as residues. The stabilizer in G of 1 is $C = G \cap H$. We may assume that $\sigma\Omega_1 = \Omega_1$, and then $H = N_\Gamma(G) = C\langle \sigma \rangle$. If we assume that $1 \in \Omega_1$, then $D = \vartheta C \vartheta^{-1}$ is the stabilizer in G of $\vartheta 1 \in \vartheta\Omega_1 = \Omega_2$ and $E = \vartheta D \vartheta^{-1}$ is the stabilizer in G of $\vartheta 2 \in \vartheta\Omega_2 = \Omega_3$, and $B = C \cap D \cap E$ is the stabilizer in G of the chamber $\{1, \vartheta 1, \vartheta 2\}$ in the associated geometry. The groups $C \cap D$, $C \cap E$ and $D \cap E$ are conjugate in Γ.

The examples are specified by giving the group G, the sets Ω_1, Ω_2 and Ω_3, and the isomorphism types of $C \cap D$ and B, and the parameters are listed. References to relevant pages of [ATLAS] are given. [CAYLEY] has been used to investigate the examples.

Example 1 is a family of examples associated with classical triality. In Examples 1 and 2 blocks are cliques in the strongly regular graph, and G acts as a flag-transitive group of automorphisms of the associated geometry. Consideration is given to these two situations in the following two sections. The parameters for Examples 1 and 2 are given so that the blocks are cliques rather than cocliques to be consistent with the general discussion in Section 6, where the formulas in this form are somewhat simpler. In Section 7, it is shown that for Example 2, the geometry is simply connected by showing (using CAYLEY) that $U_3(5)$ is the universal group of the relevant amalgam.

EXAMPLE 1 [ATLAS, pp. 85, 149].

$$G = O_8^+(q).$$

Ω_1 is the set of singular points of $PG_7(q)$, and Ω_2 and Ω_3 are the two G-orbits of maximal totally singular subspaces.

$C \cong q^6 : L_4(q)$, $C \cap D \cong q^{6+3} : L_3(q)$, $B \cong q^{6+3+2} : GL_2(q)$.

Writing $(n)_q := q^n - 1/q - 1$, the parameters are $v = (q^3+1)(4)_q$, $S = (4)_q$, $a = (2)_q$, $b = 0$, $N = q(3)_q$, $P = (3)_q$, $c = (3)_q$, $d = 1$, $k = q(q^2+1)(3)_q$, $l = q^6$, $\mu = (q^2+1)(3)_q$, $r = q^3 - 1$, $s = -(q^2+1)$, $f = q(q^2+1)^2$, $g = q^2(q^4+q^2+1)$. G acts as a flag-transitive group of automorphisms on the geometry, and the rank 2 residues are isomorphic with the symmetric 2-design of points and hyperplanes of $PG_3(q)$. We refer to the schemes of this example as the *triality schemes*.

EXAMPLE 2 [ATLAS, p. 34].

$$G = U_3(5).$$

Ω_1 is the set of 50 vertices of the complement of the Hoffman-Singleton graph, and Ω_2 and Ω_2 are the two G-orbits of maximal cliques of size 15 in this graph.

$C \cong A_7$, $C \cap D \cong L_3(2)$, $B \cong 7:3$.

v	S	a	b	N	P	c	d	k	l	λ	μ	r	s	f	g
50	15	5	0	14	12	8	3	42	7	35	36	2	-3	21	28

G acts as a flag-transitive group of automorphisms on the geometry, and the rank 2 residues are isomorphic with the complement of the symmetric 2-design of the points and hyperplanes of $PG_3(2)$.

EXAMPLE 3 [ATLAS, p. 23].

$$G = L_3(4).$$

$\Omega_1, \Omega_2, \Omega_3$ are the three G-orbits of ovals in the projective plane of order 4.

$C \cong A_6$, $C \cap D = C \cap E = D \cap E \cong 3^2 : 2$.

v	S	a	b	N	P	c	d	k	l	λ	μ	r	s	f	g
56	20	8	2	18	15	11	5	45	10	36	36	3	-3	20	35

G has three orbits of chambers, two of length $56 \cdot 20$ and one of length $56 \cdot 20 \cdot 9$.

EXAMPLE 4 [ATLAS, p. 115].

$$G = U_6(2).$$

$\Omega_1, \Omega_2, \Omega_3$ are three orbits of 222-2233-points in the Leech lattice.

$C \cong U_4(3) : 2$, $C \cap D = C \cap E = D \cap E \cong 3^{1+4} : D_8$.

v	S	a	b	N	P	c	d	k	l	λ	μ	r	s	f	g
1408	112	16	4	81	42	31	7	567	840	246	216	39	-9	252	1155

G has two orbits of chambers, one of length $1408 \cdot 112$ and one of length $1408 \cdot 112 \cdot 30$.

6. THE CASE IN WHICH BLOCKS ARE CLIQUES

In this section, we consider a uniform class I scheme with $t = 2$. In Examples 1 and 2 of Section 5, blocks of the corresponding geometry are cliques in the strongly regular graph. The condition that blocks are cliques is equivalent to any one of the conditions $N = S - 1$, $b = 0$, $S = am$, or $k = m(S - 1)$, and in that case the parameters can be given in terms of v, a and $m = -s > 0$ according to $S = am$, $N = am - 1$, $k = m(am - 1)$, $P = [am(m - 1)(am - 1)]/(v - am)$, $r = [(am - 1)(v - am^2)]/(v - am)$, $\mu = Pm$, $f = [m(v - am)^2]/X$, $u^2 = aX/(v - am)$, where $X = am(v - am^2) + v(m - 1)$, and $c = d + \varepsilon u$, $d = [am(am - \varepsilon u)]/v$.

PROPOSITION. *Assume that*

(i) *blocks are cliques, and*

(ii) *for each nonincident point x and block y, there is a unique block z incident with x and y. Then $v = (q+1)(q^2+1)(q^3+1)$, $a = q+1$, $m = q^2+1$. (Thus the parameters are those of the triality scheme in case q is a prime power.)*

Proof. The conditions (i) and (ii) mean that $b = 0$ and $d = 1$. Thus (3.1), (3.2) and (3.4) become (a) $c(c-1) = a(S-1)$, (b) $c + S - 1 = aP$, and (c) $v - S = S(S-c)$. Since $a \mid S$, (b) implies that $a \mid c-1$, so we can write $u = c - 1 = aq$ with q an integer. By (c), $S \mid v$, so $v = S\alpha$ with $\alpha > 1$ an integer, and (c) becomes $\alpha = S - c + 1$. But then $a \mid \alpha$, and writing $\alpha = a\alpha_0$, we have $S = a(q + \alpha_0)$, $m = q + \alpha_0$. By (a), $(aq+1)aq = a(a(q+\alpha_0) - 1)$, so $\alpha = a\alpha_0 = q(a(q-1)+1)+1$, and therefore $a \mid q+1$ and $(q, \alpha_0) = 1$. Now $f = [m(v-s)]/r + m = [(q+\alpha_0)^2(aq - a + 1)]/q$, so $q \mid a - 1$. It follows that $a = q+1$, $\alpha = q^3 + 1$, $m = q^2 + 1$, and $S = (q+1)(q^2+1)$. Hence by (c), $v = S\alpha = (q+1)(q^2+1)(q^3+1)$. ■

Of course, we would like to know if the conditions (i) and (ii) of the proposition characterize the triality schemes. An approach is to try to reconstruct the polar space associated with $O_8^+(q)$, and the problem is to show that the obvious "lines" have sufficiently many "points." We can observe that for $\alpha \neq \beta$, $(\Omega_\alpha, \Omega_\beta, *)$ is a partial λ-geometry, $\lambda = a$, in the sense of Cameron and Drake [CD]. By the proposition, the nexus has the appropriate value for the application of Theorem 3.6 of that paper. Then the problem becomes that of verifying that this partial a-geometry is extremal in the sense of their Definition 3.1. This we are unable to do at the present time. We obtain a characterization by assuming the condition of their Definition 3.1. (The connection with [CD] was pointed out by the referee.)

7. FLAG-TRANSITIVE CLASS I SCHEMES

In Examples 1 and 2 of Section 5, the group G acts as a flag-transitive group of automorphisms of the geometry. This means in particular that the subgroups G, C, D, E, $C \cap D$, $C \cap E$, $D \cap E$ and B form a sublattice of the lattice of subgroups of G isomorphic with the lattice of subsets of a 3-set. In each case, the rank 2 residues of the geometry are symmetric $2-(S, c, [c(c-1)]/(S-1)$ designs. For Example 1, the design is the design of points and hyperplanes of $PG_4(q)$ for which the flags are the usual ones. For Example 2, it is the design of points and hyperplanes of $PG_4(2)$ for which the flags are the anti-flags of the projective space.

Let us prove that the geometry of Example 2 is simply connected by proving that the universal group of the amalgam of the groups C, D and E is $G = U_3(5)$ in that case. In [CAYLEY] we do the calculations inside $P\Gamma U_3(5)$. We find three copies C, D and E of A_7 embedded in $G = U_3(5)$, with $C \cap D \cong C \cap E \cong D \cap E \cong L_3(2)$ and $B = C \cap D \cap E = 7:3$, and we find an automorphism ϑ of G of order 3 mapping C to D and D to E. We find elements r_1 and s generating C with $s \in B$ and $\vartheta(s) = s^2$, satisfying the relations

$$r_1^7 = s^7 = (r_1 s)^3, \qquad (r_1^{-2}s^2)^2 = (r_1^{-3}s^3)^2 = 1. \tag{a}$$

The elements $r_2 = \vartheta(r_1)$ and s generate D, and the elements $r_3 = \vartheta(r_2)$ and s generate E. Applying ϑ to the relations (a) gives the relations

$$r_2^7 = (r_2 s^2)^3, \qquad (r_2^{-2}s^4)^2 = (r_2^{-3}s^6)^2 = 1, \tag{b}$$

and applying ϑ once more gives

$$r_3^7 = (r_3 s^4)^3, \qquad (r_3^{-2}s)^2 = (r_3^{-3}s^5)^2 = 1. \tag{c}$$

Listing the elements $r_1^i s^j r_1^k$ and $r_2^i s^j r_2^k$ which are in $C \cap D$ (none of which are in B) and checking for equalities, we obtain the relations

$$r_1^3 s^4 r_1^3 = r_2^5 s^4 r_2^5, \qquad r_1^4 s^3 r_1^4 = r_2^2 s^3 r_2^2, \tag{d}$$

and applying ϑ, we get

$$r_2^3 s r_2^3 = r_3^5 s r_3^5, \qquad r_2^4 s^6 r_2^4 = r_3^2 s^6 r_3^2 \tag{e}$$

and

$$r_3^3 s^2 r_3^3 = r_1^5 s^2 r_1^5, \qquad r_3^4 s^5 r_3^4 = r_1^2 s^5 r_1^2. \tag{f}$$

Similarly, we obtain the relations

$$r_1^2 s^4 r_1^2 s^6 r_1^3 = r_2^6 s^2 r_2^4, \qquad r_1^3 s r_1^5 s^6 r_1^4 = r_2^3 s^5 r_2, \tag{g}$$

$$r_2^2 s r_2^2 s^5 r_2^3 = r_3^6 s^4 r_3^4, \qquad r_2^3 s^2 r_2^5 s^5 r_2^4 = r_2^3 s^3 r_3, \tag{h}$$

and

$$r_3^2 s^2 r_3^2 s^3 r_3^3 = r_1^6 s r_1^4, \qquad r_3^3 s^4 r_3^5 s^3 r_3^4 = r_1^3 s^6 r_1. \tag{i}$$

Now we can check on CAYLEY that an abstract group generated by r_1, r_2, r_3 and s subject to the relations (a) through (i) has order 126,000.

8. DISTANCE-REGULAR SCHEMES IN CLASS I

A symmetric association schemes is called *distance-regular* if one of its basic graphs Γ_α is distance-regular. If the scheme has rank 5, then Γ_α is distance-regular if and only if it is connected and $p_{\alpha\alpha}^{\beta} = p_{\alpha\alpha}^{\gamma} = p_{\delta\alpha}^{\beta} \cdot p_{\delta\alpha}^{\gamma} = 0$, for some permutation (β, γ, δ) of $\{1, 2, 3, 4\} - \{\alpha\}$.

Consider a distance-regular scheme in class I. We can readily check that, with the notation as in Section 2, Γ_4 cannot be distance-regular, and hence one of Γ_2, Γ_3 must be distance-regular. Since these two cases are equivalent, it suffices to assume that Γ_3 is distance-regular. There are three cases.

CASE (i). $p_{33}^1 = p_{33}^2 = p_{43}^1 \cdot p_{43}^2 = 0$. Here $N = 0$ and $S = P = 1$, so $r = -1$. Thus the scheme has imprimitive rank 3 residues, and will be considered again in section 9.

CASE (ii). $p_{33}^1 = p_{33}^4 = p_{23}^1 \cdot p_{23}^4 = 0$. Here $N = \sigma = 0$ and the intersection array is $\{tS, S-1, [(S-P)t(v-S)]/l, P; 1, [(S-1)tS]/l, S-P, tS\}$. If the scheme is anitpodal, then $P = 1$, so $r = -1$. If the scheme is bipartite, then $t = 1$ and $M_3^2 = SI + \{[(S-1)S]/l\}M_2$. But this equation implies (by examination of the $(3,3)$-entry) that $S = 1$, which is impossible.

CASE (iii). $p_{33}^2 = p_{33}^4 = p_{13}^2 \cdot p_{13}^4 = 0$. Here $S = N + 1$, $\sigma = 0$, $\rho = (t-1)S$ and the intersection array is $\{t(N+1), N, [t(N+1)(k-N)]/k; 1, [tN(N+1)]/k, P, t(N+1)\}$. If the scheme is antipodal, then $N = P$, which is impossible. If the scheme is bipartite but not antipodal, then $0 < \mu < k$, $t = 1$ and $M_3^2 = (N+1)I + \{[N(N+1)]/k\}M_1$. But this last equation is equivalent to the two equations

$$N^2 + P(k-N) = k + N\lambda, \quad \text{and} \quad Pk = N\mu. \tag{$*$}$$

The usual inequalities on the entries in the intersection array are equivalent to

$$\mu \geqslant N + 1. \tag{$**$}$$

Conclusion: The antipodal distance-regular class I schemes also belong to class II. The feasible intersection arrays for bipartite distance-regular class I schemes which are not antipodal are $\{N+1, N, [(N+1)(k-N)]/k, N+1-P; 1, [N(N+1)]/k, P, N+1\}$, where the equations $(*)$ and inequality $(**)$ hold with (v, k, λ, μ) feasible strongly regular parameters.

9. CLASS II. rank $E = 2$, corank $E = 3$

We assume that $E = \delta^{-1}(\{0, 1\})$ and write $|E(x)| = v$ for $x \in \Omega$ and $|\Omega/E| = n$, so that $|\Omega| = nv$. Let the parameters of the quotient scheme modulo E be

$$\begin{pmatrix} 0 & 1 & 0 \\ k & \lambda & \mu \\ 0 & k-\lambda-1 & k-\mu \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & k-\lambda-1 & k-\mu \\ l & \tilde{\mu} & \tilde{\lambda} \end{pmatrix},$$

$$\begin{pmatrix} 1 & k & l \\ 1 & r & -(r+1) \\ 1 & s & -(s+1) \end{pmatrix} \begin{matrix} 1 \\ f, \\ g \end{matrix} \qquad n = 1 + k + l.$$

The valencies of $\mathfrak{X}$ can be taken to have the form $v_0 = 1$, $v_1 = v - 1$, $v_2 = kS$, $v_3 = (v-k)S$, $v_4 = lv$, and the intersection matrices are then of the form

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ v-1 & v-2 & 0 & 0 & 0 \\ 0 & 0 & S-1 & S & 0 \\ 0 & 0 & v-S & v-S-1 & 0 \\ 0 & 0 & 0 & 0 & v-1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & S-1 & S & 0 \\ kS & \dfrac{kS(S-1)}{v-1} & \dfrac{(2\lambda S - \tau) - (\lambda S - \tau)v}{S} & \lambda S - \tau & \dfrac{\mu S^2}{v} \\ 0 & \dfrac{kS(v-S)}{v-1} & \dfrac{(\lambda S - \tau)(v-S)}{S} & \tau & \dfrac{\mu S(v-S)}{v} \\ 0 & 0 & (k-\lambda-1)S & (k-\lambda-1)S & (k-\mu)S \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & v-S & v-S-1 & 0 \\ 0 & \dfrac{kS(v-S)}{v-1} & \dfrac{(\lambda S-\tau)(v-S)}{S} & \tau & \dfrac{\mu S(v-S)}{v} \\ k(v-S) & \dfrac{k(v-S)(v-S-1)}{v-1} & \dfrac{\tau(v-S)}{S} & \lambda(v-S)-\tau & \dfrac{\mu(v-S)^2}{v} \\ 0 & 0 & (k-\lambda-1)(v-S) & (k-\lambda-1)(v-S) & (k-\mu)(v-S) \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & v-1 \\ 0 & 0 & (k-\lambda-1)S & (k-\lambda-1)S & (k-\mu)S \\ 0 & 0 & (k-\lambda-1)(v-S) & (k-\lambda-1)(v-S) & (k-\mu)(v-S) \\ lv & lv & \tilde{\mu}v & \tilde{\mu}v & \tilde{\lambda}v \end{pmatrix}.$$

The character-multiplicity table is

$$\begin{array}{c} \begin{pmatrix} 1 & v-1 & kS & k(v-S) & lv \\ 1 & v-1 & rS & r(v-S) & -(r+1)v \\ 1 & v-1 & sS & s(v-S) & -(s+1)v \\ 1 & -1 & x_1 & -x_1 & 0 \\ 1 & -1 & x_2 & -x_2 & 0 \end{pmatrix} \end{array} \begin{array}{l} 1 \\ f \\ g \\ z_1 \\ z_2 \end{array}$$

where x_1, x_2 are the roots of $x^2 - \{\tau v/S - \lambda(v-S)\}x - [kS(v-S)]/(v-1) = 0$. By the orthogonality relations, $kS(v-S) = -x_1x_2(v-1)$, and

$$z_1 = \frac{n(v-1)x_2}{x_2 - x_1}, \qquad z_2 = \frac{n(v-1)x_1}{x_1 - x_2}, \qquad z_1 + z_2 = n(v-1).$$

If x_1 and x_2 are irrational, then $z_1 = z_2 = [n(v-1)]/2$, $x_1^2 = x_2^2 = [kS(v-S)]/(v-1)$, and $\tau v = \lambda(v-S)$.

The Krein conditions give $(v-2)x_i^2 + x_j(v-2S) \geqslant 0$ and $x_i(kx_j - ax_i) \geqslant 0$ for $\{i, j\} = \{1, 2\}$ and $a \in \{r, s\}$.

Suppose that $E = \delta_0 \cup \delta_1$ is a class II parabolic as above and that F is a nontrivial parabolic $\neq E$. We find that if F has rank 2, then $E \vee F = \delta_0 \cup \delta_1 \cup \delta_2 \cup \delta_3$ has rank 4 and $\mathfrak{X}$ is the wreath product modulo $E \vee F$. The rank 5 schemes numbered (2), (7), (8), (9), (14) and (20) on Chang's list ([YC], Appendix A) are instances of this. If F has rank 3, then F is in class I and $E \subseteq F$. In the first case, $\mathfrak{X}$ is the wreath product modulo $E \vee F$ and in the second, $\mathfrak{X}$ is in the intersection of classes I and II. If F has rank 4, then $E \subseteq F$ and $\mathfrak{X}$ is the wreath product modulo F.

We can verify that, with notation as above, a scheme in class II is distance-transitive if and only if one of Γ_2 or Γ_3 is distance-transitive, and these two cases are equivalent under an interchange of the indices 2 and 3. We see that Γ_2 is distance-transitive if and only if $S = 1$ and $\tau = \lambda$, and in that case the intersection array is $\{k, k - \lambda - 1, [\mu(v - 1)]/v, 1; 1, \mu/v, k - \lambda - 1, k\}$. In particular, Γ_2 is antipodal when it is distance-transitive. As we will see in the next section, Γ_2 is antipodal and bipartite if and only if $\mathfrak{X}$ also belongs to class I. The examples of imprimitive distance-regular graphs of diameter 4 ([BCN, pp. 421–425]) provide examples of distance-regular class II schemes.

10. THE INTERSECTION OF CLASSES I AND II

We use the notation of Section 2. Suppose that a scheme is in class I and in class II. Then according to Section 2, we may assume that $E = f_0 \cup f_1 \cup f_2$ and $F = f_0 \cup f_1$ are parabolics. Then $\mu = 0$, $\lambda = k - 1$ and $l = t'(k + 1)$ for some integer $t' \geqslant 1$, $v = (t' + 1)(k + 1)$, and $|\Omega| = (t + 1)(t' + 1)(k + 1)$. To convert the intersection matrices from the form of Section 2 to that of Section 8, we perform the permutation $(2, 4, 3)$ on the rows and columns. Thus M_1 of Section 2 becomes

$$\hat{M}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ k & k & 0 & 0 & 0 \\ 0 & 0 & N & P & 0 \\ 0 & 0 & k - N & k - P & 0 \\ 0 & 0 & 0 & 0 & k \end{pmatrix},$$

and M_3 becomes

$$\hat{M}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & N & P & 0 \\ tS & \dfrac{NtS}{k} & \rho & \sigma & \dfrac{(S - N - 1)t(v - S)}{l} \\ 0 & \dfrac{Pt(v - S)}{k} & (t - 1)S - \rho & (t - 1)S - \sigma & \dfrac{(S - P)t(v - S)}{l} \\ 0 & 0 & S - N - 1 & S - P & 0 \end{pmatrix}.$$

Equating the first of these matrices with M_1 of Section 8, and using $\hat{}$ to indicate the parameters of Section 8, we obtain $\hat{v} = k + 1$, $\hat{S} = N + 1 = P$, and hence $r = -1$. Moreover, $S = Pv/(k - r) = [(N + 1)v]/(k + 1) =$

$\hat{S}(t'+1)$. Now equating $\hat{M}_2$ with M_2 of Section 8 gives $\hat{k}\hat{S} = tS$, so $\hat{k} = t(t'+1)$, and $\hat{k} = \hat{\mu}$ and $(\hat{k} - \hat{\lambda} - 1)S = S - N - 1$, so $\hat{\lambda} = (t-1)b$ and $\hat{l} = t'$. Moreover, $\hat{\tau} = (t-1)S - \sigma$ and $[(\hat{\lambda}\hat{S} - \hat{\tau})(\hat{v} - \hat{S})]/\hat{S} = \rho$, and from these equations and the equation $((t-1)S - \rho)S = \sigma(v - S)$ from Section 2, we obtain

$$\hat{\tau} = \frac{1}{2}\,\frac{(t-1)b(N+1)(2k-3N-1)}{a-N-1},$$

$$\rho = \tfrac{1}{2}(t-1)b(N+1), \quad \text{and} \quad \sigma = \frac{1}{2}\,\frac{(t-1)b(N+1)^2}{a-N-2}.$$

Thus the intersection matrices for a scheme in the intersection of classes I and II in the form of Section 8 are given in terms of k, N, t and t' by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ k & k-1 & 0 & 0 & 0 \\ 0 & 0 & N & N+1 & 0 \\ 0 & 0 & k-N & k-N-1 & 0 \\ 0 & 0 & 0 & 0 & k-1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & N & N+1 & 0 \\ tb(N+1) & \dfrac{tbN(N+1)}{k} & \frac{1}{2}(t-1)b(N+1) & \dfrac{1}{2}\,\dfrac{(t-1)b(N+1)}{k-N} & \dfrac{tb(N+1)^2}{k+1} \\ 0 & \dfrac{tb(N+1)(a-N-1)}{k} & \frac{1}{2}(t-1)b(N+1) & \hat{\tau} & \dfrac{tb(N+1)(k-N)}{k+1} \\ 0 & 0 & t'(N+1) & t'(N+1) & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k-N & k-N-1 & 0 \\ 0 & \dfrac{tb(N+1)(k-N)}{k} & \frac{1}{2}(t-1)b(N+1) & \hat{\tau} & \dfrac{tb(N+1)(k-N)}{k+1} \\ tb(k-N) & \dfrac{tb(k-N)(k-N-1)}{k} & \frac{1}{2}(t-1)b(2k-3N-1) & (t-1)b(k-N)-\hat{\tau} & \dfrac{tb(k-N)^2}{k+1} \\ 0 & 0 & t'(k-N) & t'(k-N) & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & k \\ 0 & 0 & t'(N+1) & t'(N+1) & 0 \\ 0 & 0 & t'(k-N) & t'(k-N) & 0 \\ t'(k+1) & t'(k+1) & 0 & 0 & (t'-1)(k+1) \end{pmatrix},$$

where $b = t' + 1$ and $\hat{\tau} = \frac{1}{2}[(t-1)(t'+1)(N+1)(2k-3N-1)]/(a-N-1)$. The character-multiplicity table is

$$\begin{pmatrix} 1 & a-1 & tb(N+1) & tb(a-N-1) & a(b-1) \\ 1 & a-1 & 0 & 0 & -a \\ 1 & a-1 & -b(a+1) & -b(a-N-1) & a(b-1) \\ 1 & -1 & x_1 & -x_1 & 0 \\ 1 & -1 & x_2 & -x_2 & 0 \end{pmatrix} \begin{array}{l} 1 \\ (t+1)(b-1) \\ t \\ z_1 \\ z_2 \end{array},$$

where x_1 and x_2 are the roots of the equation $x^2 - Bx - C = 0$, where $B = \frac{1}{2}[(t-1)b(N+1)(a-2N-2)(a-1)]/(a-N-1)$ and $C = [tb(N+1)(a-N-1)]/(a-1)$. The rank 3 quotient scheme has parameters

$$\begin{pmatrix} 0 & 1 & 0 \\ t(t'+1) & (t-1)(t'+1) & t(t'+1) \\ 0 & b-1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & t' & 0 \\ t' & 0 & t'-1 \end{pmatrix}, \quad \begin{pmatrix} 1 & tb & b-1 \\ 1 & 0 & -1 \\ 1 & -b & b-1 \end{pmatrix} \begin{array}{l} 1 \\ (t+1)t'. \\ t \end{array}$$

The intersection arrays of distance-transitive schemes in the intersection of classes I and II have the form $\{t'+1, t', [(t'+1)k]/(k+1), 1; 1, (t'+1)/(k+1), t', t'+1\}$.

11. CLASS III SCHEMES

In this case, we have a class III parabolic E, i.e., rank E = corank E = 2, and we can assume that $E = f_0 \cup f_1$. Then $|\Omega| = (t+1)v$ for some integer $t \geqslant 1$, where $|E(x)| = v$, and the valencies have the form $v_0 = 1$, $v_1 = v - 1$, $v_2 = St$, $v_3 = Tt$, $v_4 = Ut$ with $S + T + U = v$. The intersection matrices have the form

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ v-1 & v-2 & 0 & 0 & 0 \\ 0 & 0 & S-1 & S & S \\ 0 & 0 & T & T-1 & T \\ 0 & 0 & U & U & U-1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & S-1 & S & S \\ St & \dfrac{S(S-1)t}{v-1} & * & S(t-1)-\sigma-\tau & \dfrac{\rho S}{U} \\ 0 & \dfrac{STt}{v-1} & \dfrac{(S(t-1)-\sigma-\tau)T}{S} & \tau & \dfrac{\sigma T}{U} \\ 0 & \dfrac{SUt}{v-1} & \rho & \sigma & S(t-1)-\dfrac{\rho S+\sigma T}{U} \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & T & T-1 & T \\ 0 & \dfrac{STt}{v-1} & \dfrac{(S(t-1)-\sigma-\tau)T}{S} & \tau & \dfrac{\sigma T}{U} \\ Tt & \dfrac{T(T-1)t}{v-1} & \dfrac{\tau T}{S} & \varphi & \dfrac{(T(t-1)-\tau-\varphi)T}{U} \\ 0 & \dfrac{TUt}{v-1} & \dfrac{\sigma T}{S} & T(t-1)-\tau-\varphi & T(t-1)-\dfrac{(T(t-1)-\tau-\varphi+\sigma)T}{U} \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & U & U & U-1 \\ 0 & \dfrac{SUt}{v-1} & \rho & \sigma & S(t-1)-\dfrac{\rho S+\sigma T}{U} \\ 0 & \dfrac{TUt}{v-1} & \dfrac{\sigma T}{S} & T(t-1)-\tau-\varphi & T(t-1)-\dfrac{(T(t-1)-\tau-\varphi+\sigma)T}{U} \\ Ut & \dfrac{U(U-1)t}{v-1} & * & * & * \end{pmatrix}.$$

The character-multiplicity table has the form

$$\left(\begin{array}{ccccc|c} 1 & v-1 & St & Tt & Ut & 1 \\ 1 & v-1 & -S & -T & -U & t \\ 1 & -1 & a_1 & b_1 & -(a_1+b_1) & z_1, \\ 1 & -1 & a_2 & b_2 & -(a_2+b_2) & z_2 \\ 1 & -1 & a_3 & b_3 & -(a_3+b_3) & z_3 \end{array}\right)$$

$$z1 + z_2 + z_3 = (t+1)(v-1).$$

Here (a_i, b_i), $i = 1, 2, 3$, are the three distinct solutions of the system

$$A_1 a + B_1 b + C_1 = ab, \tag{11.1}$$

$$A_2 a + B_2 b + C_2 = -a(a+b), \tag{11.2}$$

$$A_3 a + B_3 b + C_3 = b^2, \tag{11.3}$$

where

$$A_1 = \frac{(S(t-1) - \sigma - \tau)T}{S} - \frac{\sigma T}{U},$$

$$B_1 = \tau - \frac{\sigma T}{U}, \qquad C_1 = -\frac{STt}{v-1},$$

$$A_2 = \rho - S(t-1) + \frac{\rho S + \sigma T}{U},$$

$$B_2 = \sigma - S(t-1) + \frac{\rho S + \sigma T}{U}, \qquad C_2 = -\frac{SUt}{v-1},$$

$$A_3 = \frac{\tau T}{S} - \frac{(T(t-1) - \tau - \varphi)T}{U},$$

$$B_3 = \varphi - \frac{(T(t-1) - \tau - \varphi)T}{U}, \qquad C_3 = \frac{(S+U)Tt}{v-1}.$$

If $a_i \neq B_1$, then by (11.1),

$$b_i = \frac{A_1 a_i + C_1}{B_1 - a}, \tag{11.4}$$

and substituting this value into (11.2) and (11.3), we see that a_i must be a root of the equations

$$a^3 + (A_1 + A_2 - B_1)a^2 + (C_2 + B_2 A_1 - A_2 B_1 + C_1)a - C_2 B_1 + B_2 C_1 = 0 \tag{11.5}$$

and

$$A_3 a^3 + \left(-2A_3 B_1 - A_1^2 + C_3 + B_3 A_1\right)a^2 + \left(B_3 C_1 + A_3 B_1^2 - B_3 A_1 B_1 - 2C_3 B_1 - 2A_1 C_1\right)a - C_1^2 - B_3 B_1 C_1 + C_3 B_1^2 = 0, \tag{11.6}$$

or equivalently, as we see by eliminating a^3 from (11.5) and (11.6), of (11.5) and

$$X_2 a^2 + X_1 a + X_0 = 0 \tag{11.7}$$

with

$$\begin{aligned} X_0 &= -A_3 C_2 B_1 + B_3 B_1 C_1 - C_3 B_1^2 + A_3 B_2 C_1 + C_1^2, \\ X_1 &= -A_3 A_2 B_1 + A_3 B_2 A_1 + 2C_3 B_1 + A_3 C_1 + B_3 A_1 B_1 \\ &\quad + A_3 C_2 - B_3 C_1 - A_3 B_1^2 + 2 A_1 C_1, \end{aligned}$$

and

$$X_2 = A_3 A_1 - C_3 - B_3 A_1 + A_1^2 + A_3 A_2 + A_3 B_1.$$

There are two cases.

Case A. $a_i \neq B_1$, $i = 1, 2, 3$.
In this case (11.5) has a_1, a_2, a_3 as distinct roots, and the b_i corresponding are given by (11.4). Since the a_i are roots of (11.7), we must have

$$X_i = 0, \qquad i = 0, 1, 2 \tag{11.8}$$

in this case.

Case B. $a_i = B_1$ for some i.
In this case by (11.1),

$$A_1 B_1 + C_1 = 0. \tag{11.9}$$

We can assume that $a_1 = B_1$, and then by (11.2) and (11.3), b_1 is a root of

$$(B_1 + B_2) b + A_2 B_1 + C_2 + B_1^2 = 0 \tag{11.10}$$

and

$$b^2 - B_3 b - A_3 B_1 - C_3 = 0. \tag{11.11}$$

CASE B.1. $a_1 = B_1$ and $B_1 + B_2 \neq 0$.
Then

$$b_1 = -\frac{A_2 B_1 + C_2 + B_1^2}{B_1 + B_2} \tag{11.12}$$

satisfies (11.11), (11.7) has the distinct roots a_2 and a_3, and b_2 and b_3 are given by (11.4).

CASE B.2. $a_1 = B_1$, $B_1 + B_2 = 0$.
In this case we must have

$$A_2 B_1 + C_2 + B_1^2 = 0 \tag{11.13}$$

and either

Case B.2.1: $a_1 = a_2 = B_1$, (11.11) has the distinct roots b_1, b_2, a_3 is the unique root of (11.7), and b_3 is given by (11.4), or

Case B.2.2: b_1 is the unique root of (11.11), a_2 and a_3 are distinct roots of (11.7), and b_2 and b_3 are given by (11.4).

The Krein conditions give

$$\frac{v}{v-1} - \frac{1}{t^2}\left(\frac{a_i^2}{S} + \frac{b_i^2}{T} + \frac{(a_i + b_i)^2}{U}\right) \geqslant 0$$

and

$$\frac{v(v-2)}{(v-1)^2} + \frac{1}{t^2}\left(\frac{a_i^2 a_j}{S^2} + \frac{b_i^2 b_j}{T^2} - \frac{(a_i + b_i)^2 (a_j + b_j)}{U^2}\right) \geqslant 0.$$

Let E be a class III parabolic. Then we find that any nontrivial parabolic $F \neq E$ must also be of class III. A family of instances of this is provided by item number (21) on Chang's list ([YC], Appendix A). This is a fusion of a pair of conference graphs of valencies k, k', respectively, which is a class III scheme having parameters as follows:

t	v	S	U	T	ρ	σ	τ	φ
$2k'$	$2k+1$	1	k	k	0	k'	$k'-1$	$kk' - \frac{k+k'}{2} + 1$

It belongs to case B.2.1). To obtain a direct description of the scheme, let $\Omega = X \times Y$, where X and Y are the vertex sets of the two conference graphs, and denote adjacency in either graph by $\sim$ and nonadjacency by $\nsim$. Define $\delta : \Omega \times \Omega \to \{0, 1, 2, 3, 4\}$ by

$$\delta((x, y), (x_1, y_1))$$

$$= \begin{cases} 0 & \text{if } x = x_1 \text{ and } y = y_1, \\ 1 & \text{if } x \neq x_1 \text{ and } y = y_1, \\ 2 & \text{if } x = x_1 \text{ and } y \neq y_1, \\ 3 & \text{if } x \sim x_1 \text{ and } y \sim y_1 \text{ or } x \nsim x_1 \text{ and } y \nsim y_1, \\ 4 & \text{if } x \sim x_1 \text{ and } y \nsim y_1 \text{ or } x \nsim x_1 \text{ and } y \sim y. \end{cases}$$

It can be verified directly that (Ω, δ) is a class III scheme by computing the intersection numbers and noting that $E = \delta_0 \cup \delta_1$ is class II parabolic. There is a second class II parabolic, namely, $F = \delta_0 \cup \delta_2$, and the corresponding class III parameters are obtained by interchanging k and k'.

REFERENCES

[ATLAS] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[BCN] A. E. Brouwer, A. M. Cohen, and A. Naimeier, *Distance Regular Graphs*, Springer-Verlag, Berlin, 1989.

[CAYLEY] J. Cannon, An introduction to the group theory language, Cayley, Computational group theory (Durham, 1982), 145–183. Academic Press, London, 1984.

[CD] P. J. Cameron and D. A. Drake, Partial λ-geometries of small nexus, *Ann. Discr. Math.* 6:19–29 (1980).

[DGH1] D. G. Higman, Coherent algebras, *Linear Algebra Appl.* 93:209–239 (1987).

[DGH2] ——, Strongly regular designs and coherent configurations of type $\begin{pmatrix} 2 & 2 \\ & 3 \end{pmatrix}$, *Eur. J. Comb.* 9:411–422 (1988).

[DGH3] ——, Parabolics of relation schemes I, preprint.

[PJC] P. J. Cameron, On groups with several doubly-transitive permutation representations, *Math. Z.* 128:1–14 (1972).

[YC] Yaotsu Chang, Primitive, symmetric association schemes of rank 4, Ph.D. thesis, University of Michigan, August 1994.

Received 24 August 1994; final manuscript accepted 13 January 1995